

CONVOLUTION AND CROSS-CORRELATION OF RAMANUJAN-FOURIER SERIES

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ABSTRACT. One of the remarkable achievements of Ramanujan[1], Hardy[2] and Carmichael[3] was the development of Ramanujan-Fourier series which converge to an arithmetic function. The Ramanujan-Fourier series, for an arithmetic function, $a(n)$, is given by

$$a(n) = \sum_{q=1}^{\infty} a_q c_q(n)$$

where the Ramanujan sum, $c_q(n)$, is defined as

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^q e^{2\pi i n \frac{k}{q}}$$

and (k, q) is the greatest common divisor of k and q .

The object of this paper is to show that if two, arithmetic functions, $a(n)$ and $b(n)$, have Ramanujan-Fourier series of,

$$a(n) = \sum_{q=1}^{\infty} a_q c_q(n) \quad \text{and} \quad b(n) = \sum_{q=1}^{\infty} b_q c_q(n)$$

then a cross-correlation between $a(n)$ and $b(n)$ can be given based on the Ramanujan-Fourier coefficients. Specifically:

$$(1) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a(n+m) \bar{b}(n) = \sum_{q=1}^{\infty} a_q \bar{b}_q c_q(m)$$

where \bar{x} is the complex conjugate of x

This paper uses the machinery of almost periodic functions to prove that even without uniform convergence the connection between a pair of almost periodic functions and the constants of the associated Fourier series exists for both the convolution and cross-correlation. The general results for two almost periodic functions are narrowed and applied to Ramanujan sums and finally applied to support the specific relation (1).

The Wiener-Khinchin formula, (1), connecting the auto-correlation of an arithmetic function and the coefficients of its Ramanujan - Fourier series is a powerful link between the circle method and the sieve methods found in number theory. The application of this Wiener-Khinchin formula to number theory is described in the works of H. G. Gadiyar and R. Padma [4], [5], [6], [7]. The Wiener-Khinchin formula is used in [4] to prove the Hardy-Littlewood conjecture and is used in [6] to prove the density of Sophie primes.

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1. INTRODUCTION

The works of Gadiyar and Padma ([4], [5], [6], and [7]) attempt to extend to arithmetic functions and their corresponding Ramanujan-Fourier series the same connection between the cross-correlation of purely periodic functions and the Fourier constants of the correlated functions. The desired result is

$$(2) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a(n+m) \bar{b}(n) = \sum_{n=1}^{\infty} a_n \bar{b}_k c_q(m)$$

for any arithmetic function with a convergent Ramanujan-Fourier series. In [4], the arithmetic functions of interest were $a(n) = \frac{\phi(n+h)}{n+h} \Lambda(n+h)$ and $b(n) = \frac{\phi(n)}{n} \Lambda(n)$. Proof of (2) for these arithmetic functions would establish the Hardy-Littlewood conjecture for primes pairs separated by an interval of h . As outlined in [6], proof of (2) would also establish the density of Sophie primes.

If the partial sums of the Ramanujan-Fourier series converge uniformly to the arithmetic function, then (2) follows easily. Unfortunately, the arithmetic functions of interest do not have a uniformly convergent Ramanujan-Fourier series.

This paper uses the machinery of almost periodic functions to prove that even without uniform convergence the connection between a pair of almost periodic functions and the constants of the associated Fourier series exists for both the convolution and cross-correlation.

The general results for two almost periodic functions are narrowed and applied to Ramanujan sums and finally applied to support the specific relation (2).

2. ALMOST PERIODIC FUNCTIONS

Besicovitch [10], Bohr [9], Corduneanu [8], and Levitan and Zhikov [11] cover many of the basic properties of almost periodic functions. From these sources on almost periodic functions, a few relevant aspects are:

$$\text{Mean-value of a function: } M \{f(x)\} = \frac{1}{2T} \int_{-T}^T f(x) dx$$

$$\text{Mean-value of a sequence: } M \{f(n)\} = \frac{1}{N} \sum_{n=1}^N f(n)$$

$$\text{Orthogonality: } M \left\{ e^{2\pi i(\lambda_k - \lambda_j)x} \right\} = \begin{cases} 1 & k = j \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Fourier series of a function: } f(x) \sim \sum_{k=1}^{\infty} a_k e^{2\pi i \lambda_k x}$$

$$\text{Fourier series of a sequence: } f(n) \sim \sum_{k=1}^{\infty} a_k e^{2\pi i \lambda_k n}$$

$$\text{Fourier constants: } a_k = M \{f(x) e^{-2\pi i \lambda_k x}\}$$

$$\text{Fourier exponents: } \lambda_k \in \mathbb{R}$$

$$\text{Uniquess Theorem: } a_k = b_k \text{ for all } k \text{ iff } f(x) = g(x)$$

Every almost periodic function generates a unique Fourier series

$$f(x) \sim \sum_{k=1}^{\infty} a_k e^{2\pi i \lambda_k x}$$

In general though, the sequence of partial sums of the Fourier series does not converge uniformly to the almost periodic function. While the Fourier series may not converge uniformly to the associated function, a related sequence of trigonometric polynomials,

$$(3) \quad \sigma_m(x) = \sum_{k=1}^{n(m)} p(k, m) a_k e^{2\pi i \lambda_k x}$$

does converge uniformly to $f(x)$ on the whole of the real line as $m \rightarrow \infty$. The sequence, $\sigma_m(x)$ are the Bochner-Fejér trigonometric polynomials described in §9 of Besicovitch [10] and Theorem 1.24 of Corduneanu [8]. The rational constants, $p(k, m)$, depend only on λ_k and m and not on a_k . The rational constants, $p(k, m)$, are in the range $0 \leq p(k, m) \leq 1$ and $\lim_{m \rightarrow \infty} p(k, m) = 1$ for all k . The partial summation is to a function of m , $n(m)$, where $\lim_{m \rightarrow \infty} n(m) = \infty$

If the Fourier constants, λ_k , are linearly independent, then $p(k, m) = 1$ and sequence of partial sums, $\sum_{k=1}^N a_k e^{2\pi i \lambda_k x}$ converge uniformly to $f(x)$ as $N \rightarrow \infty$.

(See: Theorem 1.25 p. 45[8]) If $\sum_{k=1}^{\infty} a_k e^{2\pi i \lambda_k x}$ converges uniformly to $f(x)$, then the (4a), (4b), (4c), and (4d) follow without effort.

If the λ_k are not linearly independent, then values for $p(k, m)$ are constructed from a basis for λ_k .

The goal of this paper is to apply the more general findings of almost periodic function to Ramanujan-Fourier series. Thus, the the Fourier exponents will eventually be limited to rational values. Because rational Fourier exponents are are not linearly independent, it will be assumed through out that the Fourier exponents, λ_k , are linearly dependent.

3. COMBINING ALMOST PERIODIC FUNCTIONS

Bohr [9] does consider the combining of two almost periodic functions (e.g. multiplication theorem and "folded" functions), but the assumption is that the Fourier exponents of $f(x)$ and $g(x)$ are completely arbitrary. Since, the goal of the paper is to eventually limit the Fourier exponents to the rational points on the unit interval, a considerable simplification is available when combining such functions.

Theorem 3.1. *For two almost periodic functions, $f(t)$ and $g(t)$, which share the same Fourier exponents, λ_k , and have the following Fourier series associated with the functions,*

$$f(t) \sim \sum_{k=1}^{\infty} a_k e^{2\pi i \lambda_k t} \quad \text{and} \quad g(t) \sim \sum_{k=1}^{\infty} b_k e^{2\pi i \lambda_k t}$$

and if the infinite sums below converge, then the continuous and discrete convolution and cross-correlation functions of $a(x)$ and $b(x)$ are given by:

$$(4a) \quad \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(\tau)g(t-\tau)d\tau = \sum_{k=1}^{\infty} a_k b_k e^{2\pi i \lambda_k t}$$

$$(4b) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(n)g(m-n) = \sum_{k=1}^{\infty} a_k b_k e^{2\pi i \lambda_k m}$$

$$(4c) \quad \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t+x)\bar{g}(t)dt = \sum_{k=1}^{\infty} a_k \bar{b}_k e^{2\pi i \lambda_k x}$$

$$(4d) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(n+m)\bar{g}(n) = \sum_{k=1}^{\infty} a_k \bar{b}_k e^{2\pi i \lambda_k m}$$

Proof. For the continuous convolution found on the left-hand side of (4a) and using (3)

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(\tau)g(t-\tau)d\tau = \\ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \lim_{m \rightarrow \infty} \sum_{k=1}^{n(m)} p(k, m) a_k e^{2\pi i \lambda_k \tau} \lim_{l \rightarrow \infty} \sum_{j=1}^{n(l)} p(j, l) b_j e^{2\pi i \lambda_j (t-\tau)} d\tau \end{aligned}$$

Since $p(k, m)$ and $p(j, l)$ depend the same set of Fourier exponents (λ_k and λ_j) and not a_k or b_j , there is no loss of generality if it is assumed $m = l$ and the two limit processes, $m \rightarrow \infty$ and $l \rightarrow \infty$ are collapsed into a single limit process.

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(\tau)g(t-\tau)d\tau = \\ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \lim_{m \rightarrow \infty} \sum_{k=1}^{n(m)} p(k, m) a_k e^{2\pi i \lambda_k \tau} \sum_{j=1}^{n(m)} p(j, m) b_j e^{2\pi i \lambda_j (t-\tau)} d\tau \end{aligned}$$

Since the convergence of both sums is uniform, the order of limits can be re-arranged

$$(5) \quad \begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(\tau)g(t-\tau)d\tau = \\ \lim_{m \rightarrow \infty} \sum_{k=1}^{n(m)} \sum_{j=1}^{n(m)} p(k, m)p(j, m) a_k b_j e^{2\pi i \lambda_j t} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{2\pi i (\lambda_k - \lambda_j) \tau} d\tau \end{aligned}$$

Applying the orthogonality condition:

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{2\pi i (\lambda_k - \lambda_j) \tau} d\tau = \begin{cases} 1 & k = j \\ 0 & \text{otherwise} \end{cases}$$

simplifies (5) to

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(\tau)g(t-\tau)d\tau = \lim_{m \rightarrow \infty} \sum_{k=1}^{n(m)} p(k, m)^2 a_k b_k e^{2\pi i \lambda_k t}$$

Finally, noting $\lim_{m \rightarrow \infty} p(k, m)^2 = 1$ and $\lim_{m \rightarrow \infty} n(m) = \infty$ yields:

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(\tau)g(t-\tau)d\tau = \sum_{k=1}^{\infty} a_k b_k e^{2\pi i \lambda_k t}$$

□

The proofs of (4b), (4c), and (4d) proceed in the same manner and use the uniform convergence of (3) to exchange the order of the limit processes.

While the uniform convergence of the Bochner-Fejér trigonometric polynomials was used to prove (4a), (4b), (4c), and (4d), there is no guarantee the sequence of partial summations of the sums on the right hand sides of (4a), (4b), (4c), and (4d), converge uniformly to the convolution or cross-correlation functions found on the left hand sides of (4a), (4b), (4c), and (4d). It is more precise to say the sums on the right hand sides *sum* to the functions on the left hand sides than it is to say the sums on the right hand sides *converge* to the functions on the left hand sides. This means the right and left hand sides of (4a), (4b), (4c), and (4d) are equal in a very narrow sense. If a finite limit exists, then the left and right hand sides of (4a), (4b), (4c), and (4d), have same value for this finite limit. For the goals of this paper though, this narrow sense of equal is sufficient.

4. RAMANUJAN SUMS AND RAMANUJAN-FOURIER SERIES ARE ALMOST PERIODIC FUNCTIONS

The previous section considered almost periodic functions where the Fourier exponents, λ_k , are assumed to be arbitrary reals. The Ramanujan-Fourier series

$$a(n) = \sum_{q=1}^{\infty} a_q c_q(n)$$

can be extended to the whole of the real line

$$a(x) = \sum_{q=1}^{\infty} a_q c_q(x)$$

by extending Ramanujan sums $c_q(n)$ to the real line;

$$c_q(x) = \sum_{\substack{k=1 \\ (k,q)=1}}^q e^{2\pi i \frac{k}{q} x}$$

Extended to the real line, the Ramanujan sums, $c_q(x)$ are almost periodic functions where the Fourier exponents, λ_k , are the rational points of the unit interval, $(0 \dots 1]$. The first few λ 's are:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \dots$$

If the Ramanujan-Fourier constants are: $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \dots$, then the Fourier constants corresponding to the enumerated λ 's are:

$$a_1, a_2, a_3, a_3, a_4, a_4, a_5, a_5, a_5, a_5, a_6, a_6, a_7, a_7, a_7, a_7, a_7, a_7, a_8, a_8, a_8, a_8, \dots$$

The definition of almost periodic given by Bohr [9] is that a function, $f(x)$ is almost periodic if there exist for every $\epsilon > 0$ an interval of length, L_ϵ , such that

$$(6) \quad |f(x + L_\epsilon) - f(x)| < \epsilon \quad \text{for all real } x$$

Along with uniform continuity, the definition of almost periodic given in (6), insures the uniform approximation of $f(x)$ by a sequence of Bochner-Fejér trigonometric polynomials. All purely periodic functions are also almost periodic functions. The sum, difference, or product of two, almost periodic functions is also almost periodic. Any particular term of a Ramanujan-Fourier series of $a(x)$ is purely periodic with a period of q because the Fourier exponent, $\frac{k}{q}$ is rational. The Ramanujan-Fourier series as a whole is almost periodic because, it is possible to choose length of L_ϵ as the least common multiple of the denominators, q 's, of the rational, Fourier constants. With such a translation length, $|a(x + L_\epsilon) - a(x)| = 0$ and is thus smaller than any ϵ chosen. The resulting function, $a(x)$ is uniformly continuous.

All Ramanujan-Fourier series (both finite and infinite) are almost periodic functions in the sense of Bohr [9].

Because the arithmetic function, $a(x)$,

$$(7) \quad a(x) = \sum_{q=1}^{\infty} a_q c_q(x)$$

is almost periodic and uniformly continuous, there exists a sequence of Bochner-Fejér trigonometric polynomials

$$(8) \quad \sigma_m(x) = \sum_{q=1}^{n(m)} \sum_{\substack{j=1 \\ (j,q)=1}}^q p(j, q, m) a_q e^{2\pi i \frac{j}{q} x}$$

which converges uniformly to $a(x)$ on the whole of the real line as $m \rightarrow \infty$. The constants, $p(j, q, m)$, are the rational constants defined in (3) where $\lambda_k = \frac{j}{q}$. The constants $p(j, q, m)$ have the following properties in common with $p(k, m)$.

$$\begin{aligned} 0 &\leq p(j, q, m) \leq 1 \\ \lim_{m \rightarrow \infty} p(j, q, m) &= 1 \quad \text{for all } j \text{ and } q \\ p(j, q, m) &\in \mathbb{Q} \end{aligned}$$

and the value of $p(j, q, m)$ depends only on j , q , and m and does not depend on a_q .

Theorem 4.1. *For the functions, $a(x)$ and $b(x)$, and the Ramanujan-Fourier series associated with the functions,*

$$a(x) = \sum_{q=1}^{\infty} a_q c_q(x) \quad \text{and} \quad b(x) = \sum_{q=1}^{\infty} b_q c_q(x)$$

and if the infinite sums below converge, then the convolution and cross-correlation of $a(x)$ and $b(x)$ are given by:

$$(9a) \quad \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T a(\tau) b(t - \tau) d\tau = \sum_{q=1}^{\infty} a_q b_q c_q(t)$$

$$(9b) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a(n) b(m - n) = \sum_{q=1}^{\infty} a_q b_q c_q(m)$$

$$(9c) \quad \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T a(t + x) \bar{b}(t) dt = \sum_{q=1}^{\infty} a_q \bar{b}_q c_q(x)$$

$$(9d) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a(n + m) \bar{b}(n) = \sum_{q=1}^{\infty} a_q \bar{b}_q c_q(m)$$

Proof. Beginning with the the continuous convolution on the left hand side of (9a) and using (8) yields:

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T a(\tau) b(t - \tau) d\tau = \\ & \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \lim_{m \rightarrow \infty} \sum_{q=1}^{n(m)} \sum_{\substack{k=1 \\ (k,q)=1}}^q p(k, q, m) a_q e^{2\pi i \frac{k}{q} \tau} \lim_{l \rightarrow \infty} \sum_{r=1}^{n(l)} \sum_{\substack{j=1 \\ (j,r)=1}}^r p(j, r, l) b_r e^{2\pi i \frac{j}{r} (t-\tau)} d\tau \end{aligned}$$

As before, $p(k, q, m)$ and $p(j, r, l)$ depend the same set of Fourier exponents ($\frac{k}{q}$ and $\frac{j}{r}$) and not on a_q or b_q . There is no loss of generality if it is assumed $m = l$ and the two limit processes, $m \rightarrow \infty$ and $l \rightarrow \infty$ are collapsed into a single limit process.

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T a(\tau) b(t - \tau) d\tau = \\ & \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \lim_{m \rightarrow \infty} \sum_{q=1}^{n(m)} \sum_{\substack{k=1 \\ (k,q)=1}}^q p(k, q, m) a_q e^{2\pi i \frac{k}{q} \tau} \sum_{r=1}^{n(m)} \sum_{\substack{j=1 \\ (j,r)=1}}^r p(j, r, m) b_r e^{2\pi i \frac{j}{r} (t-\tau)} d\tau \end{aligned}$$

The uniform convergence of (8) allows the order of the limit processes to be exchanged.

$$\begin{aligned} (10) \quad & \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T a(\tau) b(t - \tau) d\tau = \\ & \lim_{m \rightarrow \infty} \sum_{q=1}^{n(m)} \sum_{r=1}^{n(m)} \sum_{\substack{k=1 \\ (k,q)=1}}^q \sum_{\substack{j=1 \\ (j,r)=1}}^r p(k, q, m) p(j, r, m) a_q b_r e^{2\pi i \frac{j}{r} t} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{2\pi i (\frac{k}{q} - \frac{j}{r}) \tau} d\tau \end{aligned}$$

The orthogonality relation,

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{2\pi i \left(\frac{k}{q} - \frac{j}{r} \right) \tau} d\tau = \begin{cases} 1 & \text{if and only if } q = r \text{ and } k = j \\ 0 & \text{otherwise} \end{cases}$$

simplifies (10) to

$$(11) \quad \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T a(\tau) b(t - \tau) d\tau = \lim_{m \rightarrow \infty} \sum_{q=1}^{n(m)} \sum_{\substack{k=1 \\ (k,q)=1}}^q p(k, q, m)^2 a_q b_q e^{2\pi i \frac{k}{q} t}$$

Using the definition, $c_q(t) = \sum_{\substack{k=1 \\ (k,q)=1}}^q e^{2\pi i \frac{k}{q} t}$, simplifies (11) to

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T a(\tau) b(t - \tau) d\tau = \lim_{m \rightarrow \infty} \sum_{q=1}^{n(m)} p(k, q, m)^2 a_q b_q c_q(t)$$

Finally, applying the limit process, $\lim_{m \rightarrow \infty} p(k, q, m)^2 = 1$ and $\lim_{m \rightarrow \infty} n(m) = \infty$, brings us to

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T a(\tau) b(t - \tau) d\tau = \sum_{q=1}^{\infty} a_q b_q c_q(t)$$

□

The proofs of (9b), (9c), and (9d) proceed in the same manner.

Again, the the same warning regarding the convergence of the partial sums of (4a), (4b), (4c), and (4d) applies to the partial sums of (9a), (9b), (9c), and (9d). There is no guarantee the sequence of partial sums on the right hand sides converge uniformly to the functions on the left hand sides. But, if a finite limit exists, then left and right hand sides of (9a), (9b), (9c), and (9d), converge to the same, finite limit.

5. CONCLUSION

The discrete cross correlation, (9d), becomes the auto-correlation of $a(n)$ when $a(n) = b(n)$. Thus, the Wiener-Khinchin formula extends to almost periodic functions having the same Fourier exponents and extends to arithmetic functions with a Ramanujan-Fourier series. If finite limits exist, then the auto-correlation of an arithmetic function, $a(x)$ is related its Ramanujan-Fourier series by:

$$(12) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a(n+m) \bar{a}(n) = \sum_{q=1}^{\infty} |a_q|^2 c_q(m)$$

This is the connective link needed to complete the work discovered by Gadiyar and Padma in [4], [5], [6], and [7]. As pointed out in [7], the existence of the Wiener-Khinchin formula for arithmetic functions with Ramanujan-Fourier series provides a powerful framework in which to examine the asymptotic behavior of sieves found in various branches of number theory.

Using (12), Gadiyar and Padma prove the Hardy-Littlewood conjecture in [4]. For every even gap, $2k$ where $k > 0$, there exist an infinite number of pairs, n and

$n + 2k$, such that both n and $n + 2k$ are prime. The famous twin prime conjecture is the special case of this conjecture when $k = 1$. Using (12), Gadiyar and Padma prove similar Hardy-Littlewood conjecture regarding Sophie primes in [6]. Given two numbers, n and $2n + 1$, there are an infinite number of such pairs such that both n and $2n + 1$ are prime.

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